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Enhanced binding and mass renormalization of nonrelativistic QED¹

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Abstract

The Pauli-Fierz Hamiltonian of the nonrelativistic QED is defined as a self-adjoint operator H_Λ with ultraviolet cutoff $\Lambda > 0$, which describes an interaction between an electron and photons with momentum $< \Lambda$. Spectral properties of H_Λ are investigated for a sufficiently large Λ . In particular enhanced binding, stability of matter and asymptotic behavior of effective mass for $\Lambda \rightarrow \infty$ are studied.

1 The Pauli-Fierz Hamiltonian

This is a joint work with Herbert Spohn [20, 21].³ We consider spectral properties of a system of one spinless electron minimally coupled to a quantized radiation field quantized in the Coulomb gauge. The system is called the Pauli-Fierz model [26]. The Pauli-Fierz Hamiltonian with ultraviolet cutoff Λ is defined as a self-adjoint operator on a Hilbert space. In this paper we analyze the Hamiltonian for a sufficiently large Λ .

Since a photon is a transversely polarized wave, one particle state space of a photon is defined by $L^2(\mathbb{R}^3 \times \{1, 2\})$. Here $\mathbb{R}^3 \times \{1, 2\} \ni (k, j)$ expresses momentum and transversal component of one photon, respectively. The Boson Fock space \mathcal{F} describing a state space of photons is defined by

$$\begin{aligned} \mathcal{F} &= \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\}) \right] \\ &= \{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} \mid \Psi^{(n)} \in \bigotimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\}), \|\Psi\|^2 = \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \}, \end{aligned}$$

where $\bigotimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\})$, $n \geq 1$, denotes the n -fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1, 2\})$ and we set

$$\bigotimes_s^0 L^2(\mathbb{R}^3 \times \{1, 2\}) = \mathbb{C}.$$

The creation operator $a^*(f)$ smeared by $f \in L^2(\mathbb{R}^3 \times \{1, 2\})$ is defined by

$$(a^*(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}),$$

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where S_n denotes the symmetrization operator, i.e.,

$$S_n[\otimes^n L^2(\mathbb{R}^3 \times \{1, 2\})] = \otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\}).$$

The annihilation operator is given by

$$a(f) = (a^*(\bar{f}))^*|_{\mathcal{F}_0},$$

where \mathcal{F}_0 denotes the finite particle subspace of \mathcal{F} . Formally we often write $a^\sharp(f)$ as

$$a^\sharp(f) = \sum_{j=1,2} \int f(k, j) a^\sharp(k, j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1, 2\}).$$

Note that we do not give any rigorous mathematical meaning to formal kernel $a^\sharp(k, j)$ in this paper. $a^\sharp(k, j)$ is just a symbol. $a^\sharp(f)$ satisfy CCR,

$$\begin{aligned} [a(f), a^*(g)] &= (f, g), \\ [a(f), a(g)] &= 0, \\ [a^*(f), a^*(g)] &= 0. \end{aligned}$$

We see that

the linear hull of $\{a^*(f_1) \cdots a^*(f_n) \Omega, \Omega | f_j \in L^2(\mathbb{R}^3 \times \{1, 2\}), 1 \leq j \leq n, n \geq 1\}$

is dense in \mathcal{F} . The free Hamiltonian H_f of \mathcal{F} is defined by

$$\begin{aligned} H_f \Omega &= 0, \\ H_f a^*(f_1) \cdots a^*(f_n) \Omega &= \sum_{j=1}^n a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n) \Omega, \\ f_j &\in D(\omega), \quad j = 1, \dots, n, \end{aligned}$$

and which is formally written as

$$H_f = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk,$$

where the dispersion relation is given by

$$\omega(k) = |k|.$$

Let us denote the spectrum (resp. discrete spectrum, point spectrum, essential spectrum) of self-adjoint operator T by $\sigma(T)$ (resp. $\sigma_{\text{disc}}(T)$, $\sigma_p(T)$, $\sigma_{\text{ess}}(T)$). It is well known that

$$\sigma(H_f) = [0, \infty), \quad \sigma_p(H_f) = \{0\}.$$

Inequalities

$$\begin{aligned}\|a(f)\Psi\| &\leq \|f/\sqrt{\omega}\| \|H_f^{1/2}\Psi\|, \\ \|a^*(f)\Psi\| &\leq \|f/\sqrt{\omega}\| \|H_f^{1/2}\Psi\| + \|f\|\Psi\|\end{aligned}$$

are well known. The Pauli-Fierz Hamiltonian H is defined as a self-adjoint operator acting on

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx \quad (1.1)$$

by

$$H = \frac{1}{2m}(p_x \otimes 1 - eA_{\hat{\varphi}})^2 + V \otimes 1 + 1 \otimes H_f,$$

where $\int_{\mathbb{R}^3}^{\oplus} \cdots dx$ denotes a constant fiber direct integral, m and e the mass and the charge of electron, respectively,

$$p_x = \left(-i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2}, -i \frac{\partial}{\partial x_3} \right)$$

and V an external potential. We regard e as a coupling constant. Under identification (1.1), quantized radiation field $A_{\hat{\varphi}}$ is defined by

$$A_{\hat{\varphi}} = \int_{\mathbb{R}^3}^{\oplus} A_{\hat{\varphi}}(x) dx,$$

where

$$A_{\hat{\varphi}}(x) = \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}} e(k, j) \left\{ e^{-ikx} a^*(k, j) + e^{ikx} a(k, j) \right\} dk,$$

and, $e(k, 1)$, $e(k, 2)$ and $k/|k|$ form a three dimensional right-handed orthonormal system, i.e.,

$$e(k, j) \cdot k = 0, \quad e(k, i) \cdot e(k, j) = \delta_{ij}, \quad e(k, 1) \times e(k, 2) = k/|k|. \quad (1.2)$$

Note that

$$e(-k, 1) = -e(k, 1), \quad e(-k, 2) = e(k, 2).$$

Finally $\hat{\varphi}$ denotes a form factor. $A_{\hat{\varphi}}$ acts for $\Psi \in \mathcal{H}$ as

$$(A_{\hat{\varphi}}\Psi)(x) = A_{\hat{\varphi}}(x)\Psi(x), \quad x \in \mathbb{R}^3.$$

By (1.2), we have

$$p_x \cdot A_{\hat{\varphi}}(x) = 0.$$

The decoupled Hamiltonian is given by H with e replaced by 0, i.e.,

$$H_0 = \left(\frac{1}{2m} p_x^2 + V \right) \otimes 1 + 1 \otimes H_f.$$

Theorem 1.1 Assume that $\hat{\varphi}/\omega, \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and V is relatively bounded with respect to p_x^2 with a relative bound < 1 . Then, for arbitrary values of e , H is self-adjoint on $D(p_x^2 \otimes 1) \cap D(1 \otimes H_f)$ and bounded from below. Moreover it is essentially self-adjoint on any core of $D(H_0)$.

Proof: See [15, 16]. □

Note that

$$D(H_0) = D(p_x^2 \otimes 1) \cap D(1 \otimes H_f).$$

Quantized radiation field A_Λ with a sharp ultraviolet cutoff is defined by A_φ with $\hat{\varphi}$ replaced by

$$\chi_\Lambda(k) = \begin{cases} 0, & |k| < \kappa, \\ 1/\sqrt{(2\pi)^3}, & \kappa \leq |k| \leq \Lambda, \\ 0, & |k| > \Lambda. \end{cases}$$

Here $\kappa > 0$ is called infrared cutoff, and which is fixed throughout this paper. Hence the Hamiltonian under consideration is

$$H_\Lambda = \frac{1}{2m}(p_x \otimes 1 - eA_\Lambda)^2 + V \otimes 1 + 1 \otimes H_f.$$

In this paper we will review recent advances in analysis of the spectral properties of H_Λ for sufficiently large Λ . In particular we will discuss 1.-3.

1. Enhanced binding for a sufficiently large Λ .
2. Stability of matter as $\Lambda \rightarrow \infty$.
3. The asymptotic behavior of an effective mass as $\Lambda \rightarrow \infty$.

2 Enhanced binding

It is proven that, if $\frac{1}{2m}p_x^2 + V$ has a ground state, then H_Λ has a ground state and it is unique, under suitable conditions on V and e . See e.g., [1, 3, 8, 12, 13, 14]. We want to show, however, the existence of a ground state *without* assumption “if $\frac{1}{2m}p_x^2 + V$ has a ground state”. On a formal level we expect that bare mass m of an electron amounts to effective mass m_{eff} by a coupling with a quantized radiation field, i.e.,

$$m \rightarrow m_{\text{eff}} = m_{\text{eff}}(\Lambda) = m + \delta m(\Lambda)$$

Roughly speaking, H_Λ may be replaced by

$$H_\Lambda \sim H_{\text{eff}} = \left(\frac{1}{2m_{\text{eff}}(\Lambda)} p_x^2 + V \right) \otimes 1 + 1 \otimes H_f + \text{remainders}. \quad (2.1)$$

Since it is expected that effective mass $m_{\text{eff}}(\Lambda)$ increases as Λ does, a ground state of H_Λ could be appear for a sufficiently large Λ even when H_0 has no

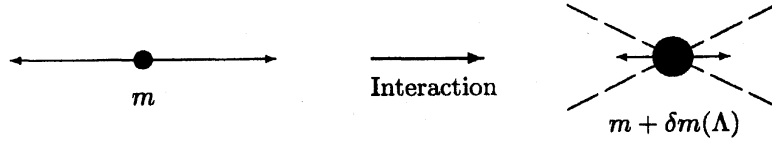


Figure 1: Effective mass

ground states. This kind of phenomena is called *enhanced binding*.

Enhanced binding for coupling constant e has been done in Hiroshima and Spohn [20], and developed by e.g., [2, 4, 5, 10]. Catto and Hainzl [4], Chen, Vougalter and Vugalter [5], Hainzl, Vougalter and Vugalter [10] study a more physically reasonable case. Arai and Kawano [2] proved the similar result as ours, i.e., enhanced binding for Λ , in a general framework.

In this section, we take the dipole approximation, i.e., $A_\Lambda(x)$ in the definition of H_Λ is replaced as

$$A_\Lambda(x) \longrightarrow 1 \otimes A_\Lambda(0).$$

Then the Hamiltonian under consideration is

$$H_{\text{dip}} = \frac{1}{2m}(p_x \otimes 1 + 1 \otimes A_\Lambda(0))^2 + V \otimes 1 + 1 \otimes H_f.$$

For notational convenience we omit the tensor notation \otimes unless confusions may arise, i.e., H_{dip} is simply written as

$$H_{\text{dip}} = \frac{1}{2m}(p_x - eA_\Lambda(0))^2 + V + H_f.$$

Assumption (V) is as follow.

Assumption V

- (a) $V \in C_0^\infty(\mathbb{R}^3)$.
- (b) $V \leq 0$.
- (c) There exists $\mu_0 > 1$ and $r > 0$ such that for $\mu > \mu_0$,

$$\inf \sigma\left(\frac{1}{2m}p_x^2 + \mu V\right) < -r.$$

Since V is relatively compact with respect to $\frac{1}{2m}p_x^2$, it holds that

$$\sigma_{\text{ess}}\left(\frac{1}{2m}p_x^2 + \mu V\right) = [0, \infty).$$

Hence $\frac{1}{2m}p_x^2 + \mu V$, $\mu > \mu_0$, has a ground state.

Remark 2.1 We do not assume the existence of ground states of $\frac{1}{2m}p_x^2 + V$.

A typical example of V is sufficiently shallow nonpositive potentials. By the Lieb-Thirring inequality [24],

$$\#\{\text{bound states of } \frac{1}{2m}p_x^2 + V\} \leq L_3 \int |mV_-(x)|^{3/2} d^3x,$$

(V_- : the negative part of V)

with some constant L_3 independent of V , we see that for a sufficiently shallow nonpositive potential V , $\frac{1}{2m}p_x^2 + V$ has no bound states. In particular it has no ground states. Thus H_0 has also no ground state.

Proposition 2.2 *There exists a unitary operator U such that*

$$U : D(p_x^2) \cap D(H_f) \rightarrow D(p_x^2) \cap D(H_f)$$

and

$$U^{-1}H_{\text{dip}}U = \frac{1}{2m_{\text{eff}}}p_x^2 + V(\cdot - K/m_{\text{eff}}) + H_f + g(\Lambda),$$

where

$$m_{\text{eff}} = m + \frac{8\pi}{3} \frac{1}{(2\pi)^3} (\Lambda - \kappa),$$

$$g(\Lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\chi_{\Lambda}/(t^2 + \omega^2)\|^2}{m + \frac{2}{3} \|\chi_{\Lambda}/\sqrt{t^2 + \omega^2}\|^2} dt,$$

and $K = (K_1, K_2, K_3)$ with

$$K_{\mu} = \sum_{j=1,2} \frac{1}{\sqrt{2}} \int \left\{ \varrho_{\mu}(k, j) a^*(k, j) + \overline{\varrho_{\mu}(k, j)} a(k, j) \right\} d^3k,$$

and $\varrho_{\mu}(\cdot, j)$ satisfies that

$$\|\omega^{n/2} \varrho_{\mu}(\cdot, j)\| \leq C \|\omega^{(n-3)/2} \chi_{\Lambda}\| \quad (2.2)$$

with some constant C .

Proof: See [20, 18]. □

We set

$$\delta V = V(\cdot - K/m_{\text{eff}}) - V,$$

$$H_{\text{eff}} = \frac{1}{2m_{\text{eff}}}p_x^2 + V,$$

and

$$\hat{H}_{\Lambda} = U^{-1}H_{\text{dip}}U = H_{\text{eff}} + \delta V + H_f + g(\Lambda).$$

Lemma 2.3 *Let Λ be such that*

$$\Lambda > (2\pi)^3 \frac{3}{8\pi} (\mu_0 - 1)m. \quad (2.3)$$

Then H_{eff} has a ground state, and

$$\#\{\text{bound states of } H_{\text{eff}}\} \leq L_3 \left(m + \frac{8\pi}{3} \frac{1}{(2\pi)^3} (\Lambda - \kappa) \right)^{3/2} \int |V(x)|^{3/2} d^3x. \quad (2.4)$$

In particular H_{eff} has a finite number of bound states.

Proof: By Hypothesis (V),

$$H_{\text{eff}} = \frac{1}{2m_{\text{eff}}} p_x^2 + V = \frac{m}{m_{\text{eff}}} \left(\frac{1}{2m} p_x^2 + \frac{m_{\text{eff}}}{m} V \right)$$

implies that if

$$\frac{m_{\text{eff}}}{m} > \mu_0, \quad (2.5)$$

then H_{eff} has a ground state. (2.5) is identical with (2.3). (2.4) follows from the Lieb-Thirring inequality. Then the lemma follows. \square

We introduce an artificial parameter $\nu > 0$, and define

$$\hat{H}_\Lambda^\nu = H_{\text{eff}} + \delta V^\nu + H_f^\nu + g(\Lambda),$$

where δV^ν and H_f^ν are defined by δV and H_f with ω replaced by $\omega + \nu$, respectively. It is easily seen that

$$\|\delta V^\nu \Psi\| \leq \theta(\Lambda) (\|H_f^{\nu 1/2} \Psi\| + \|\Psi\|)$$

with some constant $\theta(\Lambda)$ independent of ν . Actually it is presented as

$$\theta(\Lambda) = \frac{\|\nabla V\|_\infty}{m_{\text{eff}}} (\|\chi_\Lambda/\omega^2\| + \|\chi_\Lambda/\omega^{3/2}\|) \times \text{const.}$$

Note that

$$m_{\text{eff}} \sim \Lambda, \quad \|\chi_\Lambda/\omega^2\| \sim \Lambda^{1/2}, \quad \|\chi_\Lambda/\omega^{3/2}\| \sim \log \Lambda,$$

as $\Lambda \rightarrow \infty$, we have

$$\lim_{\Lambda \rightarrow \infty} \theta(\Lambda) = 0. \quad (2.6)$$

Lemma 2.4 *Suppose that $\min\{|\inf \sigma(H_{\text{eff}})|/3, 2\} > \theta(\Lambda)$. Then*

$$\sigma(\hat{H}_\Lambda^\nu) \cap [\inf \sigma(\hat{H}_\Lambda^\nu), \inf \sigma(\hat{H}_\Lambda^\nu) + \nu) \subset \sigma_{\text{disc}}(\hat{H}_\Lambda^\nu).$$

In particular \hat{H}_Λ^ν has a ground state.

Proof: See [18, Lemma 10]. □

The number operator N of \mathcal{F} is defined by

$$N = \sum_{j=1,2} \int a^*(k, j) a(k, j) d^3k.$$

I.e.,

$$(N\Psi)^{(n)} = n\Psi^{(n)},$$

$$D(N) = \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \mid \sum_{n=0}^\infty n^2 \|\Psi^{(n)}\|^2 < \infty\}.$$

A ground state of \hat{H}_Λ^ν is denoted by $\varphi_g(\nu)$.

Lemma 2.5 *Suppose that $\min\{|\inf \sigma(H_{\text{eff}})|/3, 2\} > \theta(\Lambda)$. Then, for ν such that $|\inf \sigma(H_{\text{eff}})| > 3\theta(\Lambda) + \nu$,*

$$\frac{\|N^{1/2}\varphi_g(\nu)\|}{\|\varphi_g(\nu)\|} \leq C(\max_\mu \|\nabla_\mu V\|_\infty) \frac{\|\chi_\Lambda/\omega^{5/2}\|}{m_{\text{eff}}} \quad (2.7)$$

with some constant C .

Proof: We set $E = \inf \sigma(\hat{H}_\Lambda^\nu)$. Since

$$[\hat{H}_\Lambda^\nu, a(k, j)] = -(\omega(k) + \nu)a(k, j) + [\delta V^\nu, a(k, j)],$$

we have

$$(\hat{H}_\Lambda^\nu - E + \omega(k) + \nu)a(k, j)\varphi_g(\nu) = [\delta V^\nu, a(k, j)]\varphi_g(\nu).$$

Note that

$$V(\cdot - K^\nu/m_{\text{eff}}) = e^{-i\frac{p \cdot K^\nu}{m_{\text{eff}}}} V e^{i\frac{p \cdot K^\nu}{m_{\text{eff}}}},$$

where K^ν is defined by K with ω replaced by $\omega + \nu$. Then we see that

$$[\delta V^\nu, a(k, j)] = e^{-i\frac{p \cdot K^\nu}{m_{\text{eff}}}} [V, e^{i\frac{p \cdot K^\nu}{m_{\text{eff}}}} a(k, j) e^{-i\frac{p \cdot K^\nu}{m_{\text{eff}}}}] e^{i\frac{p \cdot K^\nu}{m_{\text{eff}}}}.$$

Since

$$e^{i\frac{p \cdot K^\nu}{m_{\text{eff}}}} a(k, j) e^{-i\frac{p \cdot K^\nu}{m_{\text{eff}}}} = a(k, j) - \frac{i}{\sqrt{2m_{\text{eff}}}} p \cdot \varrho^\nu(k, j),$$

it follows that

$$[\delta V^\nu, a(k, j)] = e^{-i\frac{p \cdot K^\nu}{m_{\text{eff}}}} \left(\frac{1}{\sqrt{2m_{\text{eff}}}} (\nabla V) \cdot \varrho^\nu(k, j) \right) e^{i\frac{p \cdot K^\nu}{m_{\text{eff}}}}.$$

Thus we obtain that

$$\begin{aligned} a(k, j)\varphi_g(\nu) &= (\hat{H}_\Lambda^\nu - E + \omega(k) + \nu)^{-1} \times \\ &\times e^{-i\frac{p \cdot K^\nu}{m_{\text{eff}}}} \left(\frac{1}{\sqrt{2m_{\text{eff}}}} (\nabla V) \cdot \varrho^\nu(k, j) \right) e^{i\frac{p \cdot K^\nu}{m_{\text{eff}}}} \varphi_g(\nu). \end{aligned}$$

Using this identity we see that

$$\begin{aligned}
& (N^{1/2}\varphi_g(\nu), N^{1/2}\varphi_g(\nu)) \\
&= \sum_{j=1,2} \int \|a(k,j)\varphi_g(\nu)\|^2 d^3k \\
&= \sum_{j=1,2} \int \left\| (\hat{H}_\Lambda^\nu - E + \omega(k) + \nu)^{-1} \times \right. \\
&\quad \left. \times e^{-i\frac{p \cdot K \nu}{m_{\text{eff}}}} \left(\frac{1}{\sqrt{2}m_{\text{eff}}} (\nabla V) \varrho^\nu(k,j) \right) e^{i\frac{p \cdot K \nu}{m_{\text{eff}}}} \varphi_g(\nu) \right\|^2 d^3k \\
&\leq 3 \sum_{\mu=1}^3 \sum_{j=1,2} \int \left(\frac{1}{\omega(k)} \|\nabla_\mu V\|_\infty \right)^2 \left| \frac{1}{\sqrt{2}m_{\text{eff}}} \varrho_\mu^\nu(k,j) \right|^2 d^3k \|\varphi_g(\nu)\|^2 \\
&\leq C \left(\max_\mu \|\nabla_\mu V\|_\infty \right)^2 \frac{\|\chi_\Lambda / \omega^{5/2}\|^2}{m_{\text{eff}}^2} \|\varphi_g(\nu)\|^2.
\end{aligned}$$

Hence the lemma follows. \square

Remark 2.6 Although we used a formal calculation of $a(k,j)$ in the proof of Lemma 2.5, (2.7) can be justified in [19] rigorously.

We normalize $\varphi_g(\nu)$, i.e.,

$$\|\varphi_g(\nu)\| = 1.$$

Take a subsequence ν' such that $\varphi_g(\nu')$ weakly converges to a vector φ_g as $\nu' \rightarrow \infty$.

Proposition 2.7 Assume that $\varphi_g \neq 0$. Then φ_g is a ground state of H_{dip} .

Proof: See [1, Lemma 4.9]. \square

Theorem 2.8 There exists Λ_* such that for $\Lambda > \Lambda_*$, H_{dip} has a ground state.

Proof: It is enough to prove $\varphi_g \neq 0$ by Proposition 2.7. Let E_B denote the spectral projection of H_{eff} to a Borel set $B \subset \mathbb{R}$. Let P_Ω be the projection onto the one-dimensional subspace $\{\alpha\Omega \mid \alpha \in \mathbb{C}\}$, and we set

$$Q = E_{[\Sigma+\delta, \infty)} \otimes P_\Omega$$

with some $\delta > 0$ such that

$$\delta > \frac{3}{2}\theta(\Lambda).$$

Note that $1 \otimes N + 1 \otimes P_\Omega \geq 1$. Hence

$$E_{[\Sigma, \Sigma+\delta)} \otimes P_\Omega \geq 1 - 1 \otimes N - Q. \quad (2.8)$$

Suppose that $\min\{|\inf \sigma(H_{\text{eff}})|/3, 2\} > \theta(\Lambda)$. Then it is established in [18, Lemma 12] that

$$\frac{\|Q\varphi_g(\nu')\|}{\|\varphi_g(\nu')\|} \leq \sqrt{\frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)}} \quad (2.9)$$

for ν' such that

$$|\inf \sigma(H_{\text{eff}})| > 3\theta(\Lambda) + \nu'. \quad (2.10)$$

Then for ν' such as (2.10), we have by (2.8),

$$\begin{aligned} & (\varphi_g(\nu'), E_{[\Sigma, \Sigma+\delta)} \otimes P_\Omega \varphi_g(\nu')) \\ & \geq \|\varphi_g(\nu')\|^2 - (\varphi_g(\nu'), N\varphi_g(\nu')) - (\varphi_g(\nu'), Q\varphi_g(\nu')) \\ & = 1 - \left\{ \frac{C\|\chi_\Lambda/\omega^{5/2}\|}{m_{\text{eff}}} (\max_\mu \|\nabla_\mu V\|_\infty) \right\}^2 - \frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)}. \end{aligned}$$

Note that by (2.6),

$$\lim_{\Lambda \rightarrow \infty} \frac{\|\chi_\Lambda/\omega^{5/2}\|}{m_{\text{eff}}} = 0, \quad \lim_{\Lambda \rightarrow \infty} \frac{\theta(\Lambda)}{\delta - \frac{3}{2}\theta(\Lambda)} = 0.$$

Hence for sufficiently large Λ ,

$$(\varphi_g(\nu'), (E_{[\Sigma, \Sigma+\delta)} \otimes P_\Omega) \varphi_g(\nu')) > \epsilon$$

uniformly in ν' with some $\epsilon > 0$. Take $\nu' \rightarrow \infty$ on the both sides above. Since $E_{[\Sigma, \Sigma+\delta)} \otimes P_\Omega$ is a finite rank operator, we have

$$(\varphi_g, (E_{[\Sigma, \Sigma+\delta)} \otimes P_\Omega) \varphi_g) > \epsilon,$$

which implies $\varphi_g \neq 0$. Then φ_g is a ground state of \hat{H}_Λ . Hence H_{dip} has a ground state. \square

Remark 2.9 *The uniqueness of the ground state of H_{dip} can be also established. See [14].*

3 Stability of matter

As a corollary of Proposition 2.2 we can see a stability of matter with respect to Λ . Stability of matter investigated in this section is pointed out in e.g., Lieb and Loss [22, 23] and Fefferman, Fröhlich and Graf [7].

3.1 $g(\Lambda)/\Lambda^z$

In the case of $V = 0$, from Proposition 3.3 it follows that

$$g(\Lambda) = \inf \sigma(H_{\text{dip}}).$$

We want to see the asymptotic behavior of $g(\Lambda)$ as $\Lambda \rightarrow \infty$.

Remark 3.1 From a formal perturbation theory it follows that

$$g(\Lambda) \sim (f \otimes \Omega, H_{\text{dip}} f \otimes \Omega) = (f \otimes \Omega, \frac{1}{2m}(p_x^2 + e^2 A_\Lambda(0)^2) f \otimes \Omega) \sim \Lambda^2$$

as $\Lambda \rightarrow \infty$. As will be seen later, this is, however, incorrect.

Since

$$\|\chi_\Lambda / \sqrt{t^2 + \omega^2}\|^2 = \frac{4\pi}{(2\pi)^3} \left\{ (\Lambda - \kappa) + t \left(\tan^{-1} \frac{\kappa}{t} - \tan^{-1} \frac{\Lambda}{t} \right) \right\},$$

and

$$\begin{aligned} & \|\chi_\Lambda / (t^2 + \omega^2)\|^2 \\ &= \frac{4\pi}{(2\pi)^3} \left\{ \frac{1}{2t} \left(\tan^{-1} \frac{\Lambda}{t} - \tan^{-1} \frac{\kappa}{t} \right) + \frac{1}{2} \left(\frac{\kappa}{t^2 + \kappa^2} - \frac{\Lambda}{t^2 + \Lambda^2} \right) \right\}, \end{aligned}$$

we have

$$g(\Lambda) = 4\Lambda^2 \int_0^\infty \frac{(\tan^{-1} r - \frac{r}{1+r^2}) - (\tan^{-1} r(\frac{\kappa}{\Lambda}) - \frac{r(\frac{\kappa}{\Lambda})}{1+r^2(\frac{\kappa}{\Lambda})^2})}{(2\pi)^3 m r + \frac{8\pi}{3} \Lambda \{(r - \tan^{-1} r) - (r(\frac{\kappa}{\Lambda}) - \tan^{-1} r(\frac{\kappa}{\Lambda}))\}} \frac{dr}{r^2}.$$

In [18] the following proposition is established.

Proposition 3.2 Assume that $(2\pi)^3 m > 8\pi\kappa/3$. Then

$$\frac{8}{3} \left(\frac{3}{8\pi} \frac{1}{(2\pi)^3} \frac{1}{m} \right)^{1/2} \frac{\pi}{2} \leq \lim_{\Lambda \rightarrow \infty} \frac{g(\Lambda)}{\Lambda^{3/2}} \leq \frac{8}{3} \left(\frac{9}{8\pi} \frac{1}{(2\pi)^3} \frac{1}{m} \right)^{1/2} \frac{\pi}{2}.$$

3.2 $g(\Lambda, N)/N^z$

We consider an N particle system. We assume simply that each particle has mass m and there is no external potential. The Hamiltonian, H_{dip}^N , is defined as a self-adjoint operator acting on $L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$, and is given by

$$H_{\text{dip}}^N = \sum_{j=1}^N \frac{1}{2m} (p_j + A_{j\Lambda}(0))^2 + H_{\text{f}},$$

where

$$A_{j\Lambda}(0) = \sum_{j'=1,2} \int \frac{\chi_{j\Lambda}(k)}{\sqrt{2\omega(k)}} e(k, j') \{a^*(k, j') + a(k, j')\} d^3k.$$

Let

$$\inf \sigma(H_{\text{dip}}^N) = g(\Lambda, N).$$

We consider the two cases such as

$$(1) \chi_{j\Lambda}(k) = \chi_\Lambda(k), \quad j = 1, \dots, N,$$

(2) $\chi_{j\Lambda}$, $j = 1, \dots, N$, are characteristic functions on closed sets in \mathbb{R}^3 such as

$$\text{supp}\chi_{j\Lambda} \cap \text{supp}\chi_{i\Lambda} \cap \{0\} = \emptyset, \quad (i \neq j).$$

Intuitively (1) describes that N electrons interact each others by exchanging photons, but in (2), they do not. We expect that $g(\Lambda, N) \sim N$ for a sufficiently large N in case (2). We have a proposition.

Proposition 3.3 *In the case of (1),*

$$g(\Lambda, N) = \frac{N}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\chi_{\Lambda}/(t^2 + \omega^2)\|^2}{m + \frac{2}{3}N \|\chi_{\Lambda}/\sqrt{t^2 + \omega^2}\|^2} dt,$$

in the case of (2),

$$g(\Lambda, N) = \sum_{j=1}^N \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\chi_{j\Lambda}/(t^2 + \omega^2)\|^2}{m + \frac{2}{3} \|\chi_{j\Lambda}/\sqrt{t^2 + \omega^2}\|^2} dt.$$

Proof: See [17]. □

In the case of (1), in a similar manner as in Proposition 3.2 we can prove the following proposition.

Proposition 3.4 *We assume case (1) and $(2\pi)^3 m > 8\pi\kappa/3$. Then*

$$\frac{8}{3} \left(\frac{3}{8\pi} \frac{1}{(2\pi)^3} \frac{1}{m} \right)^{1/2} \frac{\pi}{2} \leq \lim_{\Lambda, N \rightarrow \infty} \frac{g(\Lambda, N)}{\sqrt{N}\Lambda^{3/2}} \leq \frac{8}{3} \left(\frac{9}{8\pi} \frac{1}{(2\pi)^3} \frac{1}{m} \right)^{1/2} \frac{\pi}{2}.$$

Proof: see [18]. □

In the case of (2), if we adjust $\chi_{j\Lambda}$ such as

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\chi_{j\Lambda}/(t^2 + \omega^2)\|^2}{m + \frac{2}{3} \|\chi_{j\Lambda}/\sqrt{t^2 + \omega^2}\|^2} dt = g$$

with some constant g independent of j . Then

$$g(\Lambda, N) = Ng.$$

4 Effective mass

In this section, instead of H_{dip} , we revive H_{Λ} .

4.1 Translation invariance

The momentum of the quantized radiation field is given by

$$P_f = \sum_{j=1,2} \int k a^*(k, j) a(k, j) dk$$

and the total moment by

$$P_{\text{total}} = p_x \otimes 1 + 1 \otimes P_f.$$

Let us assume that $V \equiv 0$. Then we see that

$$[H_\Lambda, P_{\text{total}\mu}] = 0, \quad \mu = 1, 2, 3.$$

Hence H_Λ and \mathcal{H} can be decomposable with respect to $\sigma(P_{\text{total}}) = \mathbb{R}^3$, i.e.,

$$\mathcal{H} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(p) dp, \quad H_\Lambda = \int_{\mathbb{R}^3}^{\oplus} H_\Lambda(p) dp.$$

Note that

$$\begin{aligned} e^{-ix \otimes P_f} P_{\text{total}} e^{ix \otimes P_f} &= p_x, \\ e^{-ix \otimes P_f} H_\Lambda e^{ix \otimes P_f} &= \frac{1}{2m} (p_x \otimes 1 - 1 \otimes P_f - e1 \otimes A_\Lambda(0)) + 1 \otimes H_f. \end{aligned}$$

From this we obtain that for each $p \in \mathbb{R}^3$,

$$\begin{aligned} \mathcal{H}(p) &\cong \mathcal{F}, \\ H_\Lambda(p) &\cong \frac{1}{2m} (p - P_f - eA_\Lambda(0)) + H_f. \end{aligned}$$

Let

$$E_{m,\Lambda}(p) = \inf \sigma(H_\Lambda(p)), \quad p \in \mathbb{R}^3.$$

Lemma 4.1 *There exist constants p_* and e_* such that for*

$$(p, e) \in \mathcal{O} = \{(p, e) \in \mathbb{R}^3 \times \mathbb{R} \mid |p| < p_*, |e| < e_*\},$$

a ground state $\varphi_g(p)$ of $H_\Lambda(p)$ exists and it is unique. Moreover $\varphi_g(p) = \varphi_g(p, e)$ is strongly analytic and $E_{m,\Lambda}(p) = E_{m,\Lambda}(p, e)$ analytic with respect to $(p, e) \in \mathcal{O}$.

Proof: See [21]. □

Remark 4.2 *Note that $E_{m,\Lambda}(p) \in \sigma_{\text{disc}}(H_\Lambda(p))$ for $(p, e) \in \mathcal{O}$ and*

$$E_{m,\Lambda}(p) = E_{m,\Lambda}(-p).$$

In what follows we assume that $(p, e) \in \mathcal{O}$. The effective mass $m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)$ is the inverse of the curvature of energy-momentum graph $(p, E(p))$ in $\mathbb{R}^3 \times \mathbb{R}$ at $p = 0$. Precisely m_{eff} is given by

$$E_{m,\Lambda}(p) - E_{m,\Lambda}(0) = \frac{1}{2m_{\text{eff}}} |p|^2 + O(|p|^3),$$

or

$$\frac{1}{m_{\text{eff}}} = \frac{1}{3} \Delta_p E_{m,\Lambda}(p, e) \big|_{p=0}.$$

Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences

$$\Lambda \rightarrow \infty, \quad m \rightarrow 0$$

such that $E_{m,\Lambda}(p) - E_{m,\Lambda}(0)$ has a nondegenerate limit. In order to find such sequences, we want to find constants

$$\beta < 0, \quad 0 < b \quad (4.1)$$

such that

$$\lim_{\Lambda \rightarrow \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = m_{\text{ph}}, \quad (4.2)$$

where m_{ph} is a given constant. It is well known that

$$\begin{aligned} \frac{m}{m_{\text{eff}}} &= 1 - \frac{2}{3} \sum_{\mu=1,2,3} \times \\ &\times \frac{(\varphi_{\mathbf{g}}(0), (P_{\mathbf{f}} + eA_{\Lambda}(0))_{\mu} (H_{\Lambda}(0) - E_{m,\Lambda}(0))^{-1} (P_{\mathbf{f}} + eA_{\Lambda}(0))_{\mu} \varphi_{\mathbf{g}}(0))}{(\varphi_{\mathbf{g}}(0), \varphi_{\mathbf{g}}(0))}. \end{aligned} \quad (4.3)$$

From this we see that m_{eff}/m is a function of e^2 , Λ/m and κ/m . Let

$$\frac{m_{\text{eff}}}{m} = f(e^2, \Lambda/m, \kappa/m).$$

To find constants (4.1), it is enough to find constants

$$0 \leq \gamma < 1, \quad 0 < b_0$$

such that

$$\lim_{\Lambda \rightarrow \infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0.$$

Actually, taking

$$\beta = \frac{-\gamma}{1-\gamma} < 0, \quad b = 1/b_1^{1/\gamma},$$

we see that

$$\lim_{\Lambda \rightarrow \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = b_0 b_1,$$

where b_1 is a parameter, which is adjusted such as

$$b_0 b_1 = m_{\text{ph}}.$$

Hence (4.2) has been established. It is seen by (4.3) that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log\left(\frac{\Lambda/m + 2}{\kappa/m + 2}\right) + O(\alpha^2), \quad (4.4)$$

where

$$\alpha = \frac{e^2}{4\pi}.$$

By (4.4) one may assume that

$$f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{\alpha(8/3\pi) + \alpha^2 c}$$

for sufficiently small α and large Λ with some constant c . Then by expanding m_{eff}/m to order α^2 one may expect that

$$f(e^2, \Lambda/m, \kappa/m) \approx 1 + \alpha \frac{8}{3\pi} \log\left(\frac{\Lambda}{m}\right) + \frac{1}{2} \alpha^2 \left(\frac{8}{3\pi} \log\left(\frac{\Lambda}{m}\right) \right)^2 + c \alpha^2 \log\left(\frac{\Lambda}{m}\right) + O(\alpha^3). \quad (4.5)$$

Hence the coefficient of α^2 may diverge as $[\log(\Lambda/m)]^2$ as $\Lambda \rightarrow \infty$. It is, however, that (4.5) is not confirmed. Instead of (4.5) we prove in this section that the coefficient of α^2 diverge as $\sqrt{\Lambda/m}$ as $\Lambda \rightarrow \infty$, i.e., there exists a constant $C > 0$ such that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log\left(\frac{\Lambda/m + 2}{\kappa/m + 2}\right) + \alpha^2 C \sqrt{\Lambda/m} + O(\alpha^3).$$

The effective mass and its renormalization have been studied from a mathematical point of view by many authors. Spohn [27] investigates the effective mass of the Nelson model [25] from a functional integral point of view. Lieb and Loss [23] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [9] computed exactly the leading order in α of the effective mass of the Pauli-Fierz Hamiltonian with spin 1/2.

4.2 Asymptotics

We split $H_\Lambda(0)$ as

$$H(0) = H_0 + eH_1 + \frac{e^2}{2} H_2,$$

where

$$\begin{aligned} H_0 &= \frac{1}{2} P_f^2 + H_f, \\ H_1 &= \frac{1}{2} (P_f \cdot A_\Lambda(0) + A_\Lambda(0) \cdot P_f), \\ H_2 &= A_\Lambda(0) \cdot A_\Lambda(0). \end{aligned}$$

Let

$$\varphi_g(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n, \quad E(0) = \sum_{n=0}^{\infty} \frac{e^{2n}}{(2n)!} E_{2n}.$$

Directly we see that

$$E_0 = E_1 = E_2 = E_3 = 0, \quad (4.6)$$

$$\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -H_0^{-1} H_2 \Omega, \quad \varphi_3 = 3H_0^{-1} H_1 H_0^{-1} H_2 \Omega. \quad (4.7)$$

Substitute (4.6) and (4.7) into formula (4.3). Then we obtain that

$$\begin{aligned} \frac{m}{m_{\text{eff}}} &= 1 - e^2 \frac{2}{3} \sum_{\mu=1}^3 \left(\Omega, A_{\mu} H_0^{-1} A_{\mu} \Omega \right) \\ &- e^4 \frac{2}{3} \sum_{\mu=1}^3 \left\{ 2 \left(\Psi_3^{\mu}, H_0^{-1} \Psi_1^{\mu} \right) + \left(\Psi_2^{\mu}, H_0^{-1} \Psi_2^{\mu} \right) - 2 \left(\Psi_2^{\mu}, H_0^{-1} H_1 H_0^{-1} \Psi_1^{\mu} \right) \right. \\ &\left. - \frac{1}{2} \left(\Psi_1^{\mu}, H_0^{-1} H_2 H_0^{-1} \Psi_1^{\mu} \right) + \left(\Psi_1^{\mu}, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^{\mu} \right) \right\} + O(e^6), \quad (4.8) \end{aligned}$$

where

$$\Psi_1^{\mu} = A_{\mu} \Omega,$$

$$\Psi_2^{\mu} = -\frac{1}{2} P_{\text{f}\mu} H_0^{-1} (A^+ \cdot A^+) \Omega,$$

$$\Psi_3^{\mu} = \frac{1}{2} \left\{ -A_{\mu} H_0^{-1} (A^+ \cdot A^+) \Omega + \frac{1}{2} P_{\text{f}\mu} H_0^{-1} (P_{\text{f}} \cdot A + A \cdot P_{\text{f}}) H_0^{-1} (A^+ \cdot A^+) \Omega \right\},$$

and

$$\begin{aligned} A^- &= \sum_{j=1,2} \int \frac{\chi_{\Lambda}(k)}{\sqrt{2\omega(k)}} e(k, j) a(k, j) dk, \\ A^+ &= \sum_{j=1,2} \int \frac{\chi_{\Lambda}(k)}{\sqrt{2\omega(k)}} e(k, j) a^*(k, j) dk. \end{aligned}$$

We compute the coefficients of e^2 and e^4 in (4.8). Let

$$\begin{aligned} \frac{1}{F_j} &= \frac{1}{r_j^2/2 + r_j}, \quad j = 1, 2, \\ \frac{1}{F_{12}} &= \frac{1}{(r_1^2 + 2r_1 r_2 X + r_2^2)/2 + r_1 + r_2}, \quad r_1, r_2 \geq 0, \quad -1 \leq X \leq 1. \end{aligned}$$

A direct calculation shows that

$$\frac{m}{m_{\text{eff}}} = 1 - \alpha a_1(\Lambda/m, \kappa/m) - \alpha^2 a_2(\Lambda/m, \kappa/m) + O(\alpha^3),$$

where

$$a_1(\Lambda/m, \kappa/m) = \frac{8}{3\pi} \log \left(\frac{\Lambda/m + 2}{\kappa/m + 2} \right)$$

and

$$a_2(\Lambda/m, \kappa/m) = \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \sum_{j=1}^6 b_j(\Lambda/m, \kappa/m), \quad (4.9)$$

$$\begin{aligned} b_1(\Lambda/m, \kappa/m) &= - \int (1 + X^2) \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}}, \\ b_2(\Lambda/m, \kappa/m) &= \int (1 + X^2) \left(\frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1 r_2 X + r_2^2}{2}, \\ b_3(\Lambda/m, \kappa/m) &= \int X(-1 + X^2) r_1 r_2 \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \left(\frac{1}{F_{12}} \right)^2, \\ b_4(\kappa/m \Lambda/m) &= - \int (1 + X^2) \frac{1}{F_1} \frac{1}{F_2}, \\ b_5(\Lambda/m, \kappa/m) &= \int (1 - X^2) \left(\frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}}, \\ b_6(\Lambda/m, \kappa/m) &= \int X(-1 + X^2) r_1 r_2 \frac{1}{F_1} \frac{1}{F_2} \frac{1}{F_{12}}, \end{aligned}$$

and

$$\int = \int_{-1}^1 dX \int_{\kappa/m}^{\Lambda/m} dr_1 \int_{\kappa/m}^{\Lambda/m} dr_2 \pi r_1 r_2.$$

The main theorem in this section is as follows.

Theorem 4.3 *There exist strictly positive constants C_{\min} and C_{\max} such that*

$$C_{\min} \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.$$

Proof: We show an outline of a proof. See [21] for details. We can prove that there exists a constant $C > 0$ such that

$$\begin{aligned} |b_j(\Lambda/m)| &\leq C[\log(\chi_\Lambda/m)]^2, \quad j = 1, 4, \\ |b_2(\Lambda/m)| &\leq C(\Lambda/m)^{1/2}, \\ |b_j(\Lambda/m)| &\leq C \log(\Lambda/m), \quad j = 3, 5, 6. \end{aligned}$$

Hence there exists a constant C_{\max} such that

$$\lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.$$

Next we can show that there exists a positive constant $\xi > 0$ such that

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,$$

which implies that there exists a constant ξ' such that

$$\xi' \leq \lim_{\Lambda \rightarrow \infty} \frac{b_2(\chi_\Lambda/m)}{\sqrt{\chi_\Lambda/m}}.$$

Thus we have

$$C_{\min} \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.$$

□

Remark 4.4 (1) $a_2(\Lambda/m, \kappa/m)/\sqrt{\Lambda/m}$ converges to a nonnegative constant as $\Lambda \rightarrow \infty$. (2) By (4.9), we can define $a_2(\Lambda/m, 0)$ since $b_j(\Lambda/m)$ with $\kappa = 0$ are finite. Moreover $a_2(\Lambda/m, 0)$ also satisfies Theorem 4.3. (3) In the case of $\kappa = 0$, Chen [6] established that $H(0)$ has a ground state $\varphi_g(0)$ but does not for $H_\Lambda(p)$ with $p \neq 0$.

4.3 Concluding remarks

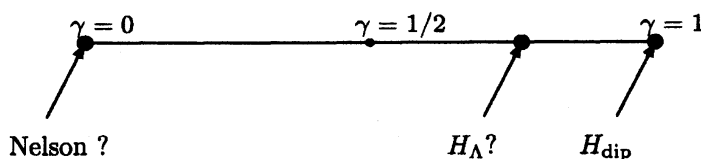


Figure 2: Mass renormalization

(H_Λ) Theorem 4.3 may suggest $\gamma \geq 1/2$ uniformly in e but $e \neq 0$.

(**Nelson models**) It is expected that the effective mass of the Nelson model can be trivially renormalized, i.e., $\gamma = 0$. See [11].

(H_{dip}) Let $V \equiv 0$. Note that

$$[H_{\text{dip}}, P_{\text{total}}] \neq 0.$$

It has been seen, however, that

$$[UH_{\text{dip}}U^{-1}, P_{\text{total}}] = 0.$$

Then we can define the effective mass m_{eff} for $UH_{\text{dip}}U^{-1}$, and which is

$$m_{\text{eff}}/m = 1 + \alpha \frac{4}{3\pi} (\Lambda/m - \kappa/m).$$

Hence $\gamma = 1$, then the mass renormalization for H_{dip} is not available.

See Fig. 2.

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